we have that

$$\mathbf{x}_2 = -\omega^2(\mathbf{T}_{12}^T\mathbf{m}_{11}\mathbf{T}_{12})^{-1}\mathbf{T}_{12}^T\mathbf{m}_{11}\mathbf{k}_{11}^{-1}\mathbf{m}_{11}\mathbf{x}_1 \quad (12)$$

Finally, substituting Eq. (12) into (11) and dividing by ω^2 we obtain the standard equation

$$(1/\omega^2)\mathbf{x}_1 = (\mathbf{I} - \mathbf{T}_{12}(\mathbf{T}_{12}^T\mathbf{m}_{11}\mathbf{T}_{12})^{-1}\mathbf{T}_{12}^T\mathbf{m}_{11})\mathbf{k}_{11}^{-1}\mathbf{m}_{11}\mathbf{x}_1 \quad (13)$$

Assuming that $\mathbf{k}_{11}^{-1}$ is available, the only matrix inversion required in Eq. (13) is for the product $(\mathbf{T}_{12}^T\mathbf{m}_{11}\mathbf{T}_{12})$ which, for three-dimensional structures, is of the order (6×6). This compares with two inversions in Eq. (9), one for $(\mathbf{T}_{12}^T\mathbf{m}_{12} + \mathbf{m}_{22})$ of order (6×6) in **R** and another for $(\mathbf{I} - \mathbf{T}_{12}\mathbf{R})$ of order $(n \times n)$, where n is the number of degrees of freedom in $\mathbf{x}_1$. It follows therefore that, in general, if the introduction of a few massless node points in the structure is an acceptable idealization so that $\mathbf{m}_{22} = \mathbf{0}$ and $\mathbf{m}_{12} = \mathbf{0}$ (zero inertia in the direction of $\mathbf{x}_2$) then it would be preferable to compute the nonzero frequencies of a free-free system from Eq. (13).

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Reply by Authors to L. Meirovitch and J. S. Przemieniecki

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THE approach presented in Meirovitch's excellent book[1] was originally developed by Mack[2] as cited in our Note. In essence, this method reduces the semidefinite free-free system to a positive definite eigenvalue problem by measuring elastic deformations of the structure from a rigid-body reference position. The singular stiffness matrix is reduced to a positive definite form by eliminating rows and columns corresponding to the rigid-body degrees of freedom and a reduced mass matrix may be obtained by either requiring that the system moments be zero or, equivalently, by imposing orthogonality of the flexible modes with the rigid-body mode shapes. However, as was stated in our note,[3] the eigenvectors of the reduced problem obtained in this manner constitute relative coordinates with respect to the assumed reference position. If the modes are required in inertial coordinates, which is usually the case in aeroelastic dynamic stability studies, an additional coordinate transformation must be performed.

Our Note circumvents this extra computation by employing a simple rigid-body transformation matrix to reduce the free-free system's stiffness and mass matrices in such a way as to obviate the need for employing relative coordinates. Thus, the flexible vibration modes are obtained directly in inertial coordinates as the eigenvectors of the reduced system.

The advantages of a symmetrical formulation, as noted in Ref. 1, are not disputed and, for this reason, were presented as Eq. (17) of our Note. We leave it to the reader to decide for himself whether "virtually the entire material presented in the Note" can be found in Ref. 1.

Turning to Przemieniecki's comments on the relative value of obtaining a nonsingular stiffness matrix, the authors agree that eigenvalue algorithms are available to accommodate singular matrices. However, their use often involves penalties with regard to accuracy or computation time. It should also be noted that many highly reliable and efficient numerical methods require the use of positive definite matrices. This fact, coupled with the increasing popularity of the stiffness method[4] (which does not deliver k_{11}^{-1} automatically) over the force method, seems to make our formulation a useful alternative to the existing ones.

Another issue introduced by Przemieniecki, following his Eq. (10), concerns the possibility of a singular mass matrix. As implied in our note, we did not consider this situation as it was assumed that m_{11} and m_{22} were positive definite. However, since this possibility exists and the point has been raised, we would suggest elimination of the noninertial degrees of freedom prior to elimination of the rigid-body modes. A standard procedure for accomplishing this is described in Ref. 5.

In connection with this latter point, it should be noted that the discussor's Eq. (13) still contains all the rigid-body modes (the matrix in braces on the right-hand side of the equation is singular). Thus, as noted earlier, Eq. (13) precludes the use of eigenvalue computational methods which require positive definite matrices.

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Comment on "The Eigenvalue Problem for Structural Systems with Statistical Properties"

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Nomenclature

K = stiffness matrix
M = mass matrix
λ_i = ith eigenvalue

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